WHEN IS THERE A NONTRIVIAL EXTENSION-CLOSED SUBCATEGORY?

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Abstract. Let R be a commutative Noetherian local ring, and denote by $\mathsf{mod}\,R$ the category of finitely generated R-modules. In this paper, we consider when $\mathsf{mod}\,R$ has a nontrivial extension-closed subcategory. We prove that this is the case if there are part of a minimal system of generators x, y of the maximal ideal with $xy = 0$, and that it holds if R is a stretched Artinian Gorenstein local ring which is not a hypersurface.

Introduction

Let R be a commutative Noetherian local ring with maximal ideal $\mathfrak{m}$. Denote by $\mathsf{mod}\,R$ the category of finitely generated R-modules. An *extension-closed* subcategory of $\mathsf{mod}\,R$ is by definition a nonempty strict full subcategory of $\mathsf{mod}\,R$ closed under direct summands and extensions. The zero R-module, the finitely generated free R-modules and all the finitely generated R-modules form extension-closed subcategories of $\mathsf{mod}\,R$, respectively. We call these three subcategories *trivial* extension-closed subcategories of $\mathsf{mod}\,R$.

In this paper, we consider when there are only trivial extension-closed subcategories and when a nontrivial one exists. In the case where R is an Artinian hypersurface, all the extension-closed subcategories of $\mathsf{mod}\,R$ are trivial. Our conjecture is that the converse also holds true.

Conjecture. The following are equivalent.

(1) R is an Artinian hypersurface.
(2) $\mathsf{mod}\,R$ has only trivial extension-closed subcategories.

Both conditions in this conjecture imply that R is an Artinian Gorenstein local ring. The conjecture holds if R is a complete intersection.

The main result of this paper is the following theorem.

Theorem. *Let x, y be part of a minimal system of generators of $\mathfrak{m}$ with $xy = 0$. Then $R/\mathfrak{m}$ does not belong to the smallest extension-closed subcategory of $\mathsf{mod}\,R$ containing $R/(x)$, and hence it is a nontrivial extension-closed subcategory.*

Let R be an Artinian local ring of length l with embedding dimension e. Recall that R is said to be *stretched* if $\mathfrak{m}^{l-e} \neq 0$. An Artinian Gorenstein local ring which is not a field and the cube of whose maximal ideal is zero is an example of a stretched Artinian Gorenstein local ring. The above theorem yields the following corollary, which guarantees that our conjecture holds when R is a stretched Artinian Gorenstein local ring.

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Corollary. *Let R be a stretched Artinian Gorenstein local ring. Then the following are equivalent.*

(1) *R is an Artinian hypersurface.*

(2) *$\mathsf{mod}\, R$ has only trivial extension-closed subcategories.*

Convention

1. Throughout the rest of this paper, we assume that all rings are commutative Noetherian local rings, and that all modules are finitely generated. Let R be a commutative Noetherian local ring. We denote by $\mathfrak{m}$ the maximal ideal of R, by k the residue field of R and by $\mathsf{mod}\, R$ the category of finitely generated R-modules.

2. Let $\mathcal{C}$ be a category. In this paper, by a *subcategory* of $\mathcal{C}$, we always mean a nonempty strict full subcategory of $\mathcal{C}$. (Recall that a subcategory $\mathcal{X}$ of $\mathcal{C}$ is said to be *strict* if every object of $\mathcal{C}$ that is isomorphic in $\mathcal{C}$ to some object of $\mathcal{X}$ belongs to $\mathcal{X}$.) By the *subcategory* of $\mathcal{C}$ consisting of objects $\{M_\lambda\}_{\lambda\in\Lambda}$, we always mean the smallest strict full subcategory of $\mathcal{C}$ to which M_λ belongs for all $\lambda \in \Lambda$. Note that this coincides with the full subcategory of $\mathcal{C}$ consisting of all objects $X \in \mathcal{C}$ such that $X \cong M_\lambda$ for some $\lambda \in \Lambda$.

3. We will often omit a letter indicating the base ring if there is no fear of confusion.

1. Some observations

We begin with recalling the precise definition of an extension-closed subcategory of $\mathsf{mod}\, R$.

Definition 1.1. Let $\mathcal{X}$ be a subcategory of $\mathsf{mod}\, R$. We say that $\mathcal{X}$ is *extension-closed* if $\mathcal{X}$ satisfies the following two conditions.

(1) $\mathcal{X}$ is closed under direct summands: if M is an R-module in $\mathcal{X}$ and N is a direct summand of M, then N is also in $\mathcal{X}$.

(2) $\mathcal{X}$ is closed under extensions: for every exact sequence $0 \to L \to M \to N \to 0$ of R-modules, if L and N are in $\mathcal{X}$, then M is also in $\mathcal{X}$.

For an R-module X, we denote by $\mathsf{add}_R X$ the *additive closure* of X, namely, the smallest subcategory of $\mathsf{mod}\, R$ containing X which is closed under finite direct sums and direct summands. This is nothing but the subcategory of $\mathsf{mod}\, R$ consisting of all direct summands of finite direct sums of copies of X. Note that the additive closure $\mathsf{add}_R R$ of R is the same as the subcategory of $\mathsf{mod}\, R$ consisting of all free R-modules.

We call the subcategory of $\mathsf{mod}\, R$ consisting of the zero R-module the *zero subcategory* of $\mathsf{mod}\, R$, and denote it by $\mathbf{0}$. Clearly,

$$\mathbf{0}, \ \mathsf{add}\, R, \ \mathsf{mod}\, R$$

are all extension-closed subcategories of $\mathsf{mod}\, R$. We call these three subcategories trivial extension-closed subcategories of $\mathsf{mod}\, R$.

Definition 1.2. We say that $\mathsf{mod}\, R$ *has only trivial extension-closed subcategories* if all the extension-closed subcategories of $\mathsf{mod}\, R$ are $\mathbf{0}$, $\mathsf{add}\, R$ and $\mathsf{mod}\, R$. If there exists an extension-closed subcategory of $\mathsf{mod}\, R$ other than these three, then we say that $\mathsf{mod}\, R$ *has a nontrivial extension-closed subcategory.*

Over an Artinian hypersurface, there exists no nontrivial extension-closed subcategory.

Proposition 1.3. *If R is an Artinian hypersurface, then $\mathsf{mod}\,R$ has only trivial extension-closed subcategories.*

Proof. This is proved in [6, Proposition 5.6]. For the convenience of the reader, we give here a proof. There exist a discrete valuation ring S with maximal ideal (x) and a positive integer n such that R is isomorphic to $S/(x^n)$. Applying to S the structure theorem for finitely generated modules over a principal ideal domain, we have

$$\mathsf{mod}\,R = \mathsf{add}_R(R \oplus R/(x) \oplus R/(x^2) \oplus \cdots \oplus R/(x^{n-1})).$$

Let $\mathcal{X}$ be an extension-closed subcategory of $\mathsf{mod}\,R$. Suppose that $\mathcal{X}$ is neither $\mathbf{0}$ nor $\mathsf{add}\,R$. Then $\mathcal{X}$ contains $R/(x^l)$ for some $1 \le l \le n-1$. For each integer $1 \le i \le n-1$ there exists an exact sequence

$$0 \to R/(x^i) \xrightarrow{f} R/(x^{i-1}) \oplus R/(x^{i+1}) \xrightarrow{g} R/(x^i) \to 0$$

of R-modules, where $x^0 := 1$, $f(\overline{a}) = \left(\begin{smallmatrix}\overline{a}\\ \overline{ax}\end{smallmatrix}\right)$ and $g(\left(\begin{smallmatrix}\overline{a}\\ \overline{b}\end{smallmatrix}\right)) = \overline{ax-b}$. Hence $\mathcal{X}$ contains both $R/(x^{l-1})$ and $R/(x^{l+1})$. An inductive argument implies that $\mathcal{X}$ contains $R/(x), R/(x^2), \dots, R/(x^{n-1}), R/(x^n) = R$. Therefore $\mathcal{X}$ coincides with $\mathsf{mod}\,R$. □

We conjecture that the converse of Proposition 1.3 also holds. The main purpose of this paper is to study this conjecture.

Conjecture 1.4. If $\mathsf{mod}\,R$ has only trivial extension-closed subcategories, then R is an Artinian hypersurface.

One can show that the assumption of Conjecture 1.4 implies that R is Artinian and Gorenstein.

Proposition 1.5. *If $\mathsf{mod}\,R$ has only trivial extension-closed subcategories, then R is an Artinian Gorenstein ring.*

Proof. First, let $\mathcal{X}$ be the subcategory of $\mathsf{mod}\,R$ consisting of all R-modules of finite length. Clearly, $\mathcal{X}$ is an extension-closed subcategory of $\mathsf{mod}\,R$. Using the fact that $\mathcal{X}$ contains k and our assumption, we easily deduce that $\mathcal{X}$ coincides with $\mathsf{mod}\,R$, which implies that R is Artinian.

Next, let $\mathcal{Y}$ be the subcategory of $\mathsf{mod}\,R$ consisting of all injective R-modules. It is obvious that $\mathcal{Y}$ is extension-closed, and the injective hull of k belongs to $\mathcal{Y}$. Our assumption implies that $\mathcal{Y}$ is equal to $\mathsf{add}\,R$, and we see that R is Gorenstein. □

In the proposition below, we give a sufficient condition for $\mathsf{mod}\,R$ to have a nontrivial extension-closed subcategory. This sufficient condition is a little complicated, but by using this, we will obtain some explicit sufficient conditions.

Proposition 1.6. *Let $S \to R$ be a homomorphism of local rings. Assume that there exist R-modules M, N such that:*

- *M is S-flat and not R-free,*
- *N is not S-flat.*

Then $\mathsf{mod}\,R$ has a nontrivial extension-closed subcategory.

Proof. Let $\mathcal{X}$ be the subcategory of $\operatorname{\mathsf{mod}} R$ consisting of all S-flat R-modules. It is easy to see that $\mathcal{X}$ is an extension-closed subcategory of $\operatorname{\mathsf{mod}} R$. The existence of M and N shows that $\mathcal{X}$ does not coincide with any of $\mathbf{0}$, $\operatorname{\mathsf{add}} R$, $\operatorname{\mathsf{mod}} R$. $\square$

The following result is a direct consequence of Proposition 1.6.

Corollary 1.7. *Suppose that there exist a local subring $S \subsetneq R$ which is not a field and an ideal $I \subsetneq R$ such that the composition $S \to R \to R/I$ is an isomorphism. Then $\operatorname{\mathsf{mod}} R$ has a nontrivial extension-closed subcategory.*

Proof. Apply Proposition 1.6 to $M = R/I$ and $N = k$. $\square$

The next three results, which give explicit sufficient conditions for $\operatorname{\mathsf{mod}} R$ to have a nontrivial extension-closed subcategory, are all deduced from Corollary 1.7.

Corollary 1.8. *Let S be a local ring which is not a field and N a nonzero S-module. Let $R = S \ltimes N$ be the idealization of N over S. Then $\operatorname{\mathsf{mod}} R$ has a nontrivial extension-closed subcategory.*

Proof. Setting $I = \{ (0, n) \in R \mid n \in N \}$, we see that the composite map $S \to R \to R/I$ of natural homomorphisms is an isomorphism. Corollary 1.7 yields the conclusion. $\square$

Corollary 1.9. *Let S, T be complete local rings which are not fields and have the same coefficient field k. Let $R = S \mathbin{\widehat{\otimes}}_k T$ be the complete tensor product of S and T over k. Then $\operatorname{\mathsf{mod}} R$ has a nontrivial extension-closed subcategory.*

Proof. We can write $S \cong k[[x_1, \dots, x_n]]/(f_1, \dots, f_a)$ and $T \cong k[[y_1, \dots, y_m]]/(g_1, \dots, g_b)$, where $n, m \geq 1$, $f_1, \dots, f_a \in (x_1, \dots, x_n)^2$ and $g_1, \dots, g_b \in (y_1, \dots, y_m)^2$. Then R is isomorphic to the ring $k[[x_1, \dots, x_n, y_1, \dots, y_m]]/(f_1, \dots, f_a, g_1, \dots, g_b)$. The composition $S \to R \to R/(y_1, \dots, y_m)R$ of natural maps is an isomorphism, and we can use Corollary 1.7. $\square$

The following result is due to Shiro Goto.

Corollary 1.10. *Let $R = k[[X_1, \dots, X_n, Y]]/\mathfrak{a}$ be a residue ring of a formal power series ring over a field k with $n \geq 1$. Assume that $Y^{l+1} \in \mathfrak{a} \subseteq (X_1, \dots, X_n, Y)^{l+1}$ holds for some $l \geq 1$. Then $\operatorname{\mathsf{mod}} R$ has a nontrivial extension-closed subcategory.*

Proof. Let $x_1, \dots, x_n, y \in R$ be the residue classes of $X_1, \dots, X_n, Y$. Let $k[[y]]$ be the k-subalgebra of R generated by y. Since $y^{l+1} = 0$, we have a surjective ring homomorphism $\phi : k[[t]]/(t^{l+1}) \to k[[y]]$ given by $\phi(\overline{f(t)}) = f(y)$ for $f(t) \in k[[t]]$, where t is an indeterminate over k. Thus we obtain a ring homomorphism

$$\psi : k[[t]]/(t^{l+1}) \xrightarrow{\phi} k[[y]] \subsetneq R \to R/(x_1, \dots, x_n) + \mathfrak{m}^{l+1} = k[[Y]]/(Y^{l+1}).$$

We see that ψ is an isomorphism. Hence ϕ is injective, and therefore it is an isomorphism. Applying Corollary 1.7 to $S = k[[y]]$ and $I = (x_1, \dots, x_n) + \mathfrak{m}^{l+1}$, we get the conclusion. $\square$

Using Corollaries 1.8 and 1.9, let us construct examples of a ring R such that $\operatorname{\mathsf{mod}} R$ has a nontrivial extension-closed subcategory.

Example 1.11. Let k be a field.

(1) Consider the ring

$$R = k[[x, y, z, w]]/(x^2, xy, xz - yw, xw, y^2, yz, z^2, zw, w^2).$$

This is an Artinian Gorenstein local ring. Putting $S = k[[x, y]]/(x^2, xy, y^2)$, we observe that R is isomorphic to the idealization $S \ltimes E_S(k)$, where $E_S(k)$ denotes the injective hull of the S-module k. Hence it follows from Corollary 1.8 that $\mathsf{mod}\, R$ has a nontrivial extension-closed subcategory.

In fact, for instance, let $\mathcal{X}$ be the subcategory of $\mathsf{mod}\, R$ consisting of all R-modules X satisfying $\mathrm{Tor}_1^R(R/(x), X) = 0$. It is clear that $\mathcal{X}$ is extension-closed. We have an exact sequence

$$0 \to R/(x, y, w) \xrightarrow{f} R \to R/(x) \to 0,$$

where $f(\overline{1}) = x$. Making the tensor product over R of this exact sequence with $R/(z)$, we get an exact sequence

$$0 \to \mathrm{Tor}_1^R(R/(x), R/(z)) \to k \xrightarrow{g} R/(z) \to R/(x, z) \to 0,$$

where $g(\overline{1}) = \overline{x}$. We see that $\mathrm{Tor}_1^R(R/(x), R/(z)) = 0$, namely, $R/(z)$ belongs to $\mathcal{X}$. Since $R/(x)$ is not a free R-module, k does not belong to $\mathcal{X}$. Thus $\mathcal{X}$ is an extension-closed subcategory of $\mathsf{mod}\, R$ which is different from any of $\mathbf{0}$, $\mathsf{add}\, R$, $\mathsf{mod}\, R$.

(2) Let

$$R = k[[x, y]]/(x^n, y^m)$$

with $n, m \geq 2$. This is an Artinian complete intersection. Since we have an isomorphism $R \cong k[[x]]/(x^n) \,\widehat{\otimes}_k\, k[[y]]/(y^m)$ of rings, $\mathsf{mod}\, R$ has a nontrivial extension-closed subcategory by Corollary 1.9.

Indeed, for example, the subcategory of $\mathsf{mod}\, R$ consisting of all R-modules X with $\mathrm{Tor}_1^R(R/(x), X) = 0$ is extension-closed, and does not coincide with any of $\mathbf{0}$, $\mathsf{add}\, R$, $\mathsf{mod}\, R$ because it contains $R/(y)$ and does not contain k.

Now, we verify that Conjecture 1.4 holds for a ring admitting a module with bounded Betti numbers.

Proposition 1.12. *Suppose that* $\mathsf{mod}\, R$ *has only trivial extension-closed subcategories. If there exists a nonfree* R*-module* M *whose Betti numbers are bounded, then* R *is an Artinian hypersurface.*

Proof. That the local ring R is Artinian follows from Proposition 1.5. Let $\mathcal{X}$ be the subcategory of $\mathsf{mod}\, R$ consisting of all R-modules whose Betti numbers are bounded. Then it is easy to see that $\mathcal{X}$ is extension-closed. Since the nonfree R-module M belongs to $\mathcal{X}$, our assumption implies that $\mathcal{X}$ coincides with $\mathsf{mod}\, R$. In particular, the module k is in $\mathcal{X}$, which forces R to be a hypersurface (cf. [7] or [1, Remarks 8.1.1(3)]). □

Using [3, Theorem 3.2], we observe that such a module M as in Proposition 1.12 exists when there exists an R-complex of finite complete intersection dimension and of infinite projective dimension. (See [2] for the details of complete intersection dimension.) Thus we obtain:

Corollary 1.13. *Assume that there exists an R-complex of finite complete intersection dimension and of infinite projective dimension. If $\operatorname{\mathsf{mod}} R$ has only trivial extension-closed subcategories, then R is an Artinian hypersurface.*

Since over a complete intersection local ring every module has finite complete intersection dimension, Corollary 1.13 and Proposition 1.5 guarantee that Conjecture 1.4 holds true in the case where the local ring R is a complete intersection. Combining this with Proposition 1.3, we get the following result.

Corollary 1.14. *If R is a complete intersection, then the following are equivalent.*

(1) *R is an Artinian hypersurface.*
(2) *$\operatorname{\mathsf{mod}} R$ has only trivial extension-closed subcategories.*

2. Main results

In this section, we conduct a closer investigation of the condition that $\operatorname{\mathsf{mod}} R$ has a nontrivial extension-closed subcategory. Establishing a certain assumption on the ring R, we shall construct an explicit nontrivial extension-closed subcategory. For this purpose, we begin with introducing a notion of a subcategory constructed from a single module.

Definition 2.1. Let X be a nonzero R-module. We define the subcategory $\operatorname{\mathsf{filt}}_R^n X$ of $\operatorname{\mathsf{mod}} R$ inductively as follows.

(1) Let $\operatorname{\mathsf{filt}}_R^1 X$ be the subcategory consisting of X.
(2) For $n \geq 2$, let $\operatorname{\mathsf{filt}}_R^n X$ be the subcategory consisting of all R-modules M such that there are exact sequences
$$0 \to Y \to M \to X \to 0$$
of R-modules with $Y \in \operatorname{\mathsf{filt}}_R^{n-1} X$.

We denote by $\operatorname{\mathsf{filt}}_R X$ the subcategory of $\operatorname{\mathsf{mod}} R$ consisting of all R-modules M such that $M \in \operatorname{\mathsf{filt}}_R^n X$ for some $n \geq 1$.

Here is a result concerning the structure of $\operatorname{\mathsf{filt}}_R^n X$. Its name comes from its property stated in the first assertion.

Proposition 2.2. *Let X be a nonzero R-module.*

(1) *An R-module M belongs to $\operatorname{\mathsf{filt}}_R^n X$ if and only if there exists a filtration*
$$0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_n = M$$
of R-submodules of M with $M_i/M_{i-1} \cong X$ for all $1 \leq i \leq n$.
(2) *If $\operatorname{\mathsf{filt}}_R^p X$ intersects $\operatorname{\mathsf{filt}}_R^q X$, then $p = q$.*

Proof. (1) This can be proved by induction on n.

(2) It is seen from the definition that if an R-module M belongs to $\operatorname{\mathsf{filt}}_R^n X$, then we have $\mathrm{e}(M) = n \cdot \mathrm{e}(X)$, where $\mathrm{e}(-)$ denotes the multiplicity. The assertion immediately follows from this. $\square$

Corollary 2.3. *Let X be a nonzero R-module.*

(1) *Let $0 \to L \to M \to N \to 0$ be an exact sequence of R-modules. If L is in $\mathsf{filt}_R^p X$ and N is in $\mathsf{filt}_R^q X$, then M is in $\mathsf{filt}_R^{p+q} X$.*

(2) *The subcategory $\mathsf{filt}_R X$ of $\mathsf{mod}\, R$ is closed under extensions.*

Proof. (1) Using Proposition 2.2(1), we can prove the assertion.

(2) This assertion follows from (1). $\square$

For an R-module X, we denote by $\mathsf{ext}_R X$ the *extension closure* of X, that is, the smallest extension-closed subcategory of $\mathsf{mod}\, R$ containing X. One can describe $\mathsf{ext}_R X$ by using $\mathsf{filt}_R X$.

Proposition 2.4. *Let X be a nonzero R-module. Then $\mathsf{ext}_R X$ coincides with the subcategory of $\mathsf{mod}\, R$ consisting of all direct summands of modules in $\mathsf{filt}_R X$.*

Proof. Let $\mathcal{X}$ be the subcategory of $\mathsf{mod}\, R$ consisting of all direct summands of modules in $\mathsf{filt}_R X$. It suffices to prove the following two statements.

(1) $\mathcal{X}$ is an extension-closed subcategory of $\mathsf{mod}\, R$ containing X.

(2) If $\mathcal{X}'$ is an extension-closed subcategory of $\mathsf{mod}\, R$ containing X, then $\mathcal{X}'$ contains $\mathcal{X}$.

As to (1): Obviously, $\mathcal{X}$ contains X and is closed under direct summands. Let $0 \to L \to M \to N \to 0$ be an exact sequence of R-modules with $L, N \in \mathcal{X}$. Then we have isomorphisms $L \oplus L' \cong Y$ and $N \oplus N' \cong Z$ for some $L', N' \in \mathsf{mod}\, R$ and $Y, Z \in \mathsf{filt}\, X$. Taking the direct sum of the above exact sequence with the exact sequences $0 \to L' \xrightarrow{=} L' \to 0 \to 0$ and $0 \to 0 \to N' \xrightarrow{=} N' \to 0$, we get an exact sequence

$$0 \to Y \to L' \oplus M \oplus N' \to Z \to 0.$$

Since Y, Z are in $\mathsf{filt}\, X$, so is $L' \oplus M \oplus N'$, and hence M belongs to $\mathcal{X}$. Thus $\mathcal{X}$ is closed under extensions.

As to (2): Since $\mathcal{X}'$ is closed under direct summands, we have only to prove that $\mathcal{X}'$ contains $\mathsf{filt}\, X$, equivalently, that $\mathcal{X}'$ contains $\mathsf{filt}^n X$ for every $n \geq 1$. This can easily be shown by induction on n. $\square$

Let x be an element of R. To understand the subcategory $\mathsf{ext}_R(R/(x))$, we investigate the structure of each module in $\mathsf{filt}_R^n(R/(x))$ for $n \geq 1$.

Proposition 2.5. *Let $x \in R$ and $n \geq 1$. Let M be an R-module in $\mathsf{filt}_R^n(R/(x))$. Then there exists an exact sequence*

$$R^n \xrightarrow{\begin{pmatrix} x & c_{1,2} & \cdots & c_{1,n} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & c_{n-1,n} \\ 0 & \cdots & 0 & x \end{pmatrix}} R^n \longrightarrow M \longrightarrow 0$$

of R-modules with each $c_{i,j}$ being in R such that

$$\begin{pmatrix} c_{1,j} \\ \vdots \\ c_{j-1,j} \end{pmatrix} (0 :_R x) \subseteq \operatorname{Im} \begin{pmatrix} x & c_{1,2} & \cdots & c_{1,j-1} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & c_{j-2,j-1} \\ 0 & \cdots & 0 & x \end{pmatrix}$$

for all $2 \leq j \leq n$.

Proof. We prove the proposition by induction on n. When $n = 1$, we have $M \cong R/(x)$, and there is an exact sequence $R \xrightarrow{x} R \to M \to 0$. Let $n \geq 2$. We have an exact sequence $0 \to Y \to M \to R/(x) \to 0$ of R-modules with $Y \in \mathsf{filt}^{n-1}(R/(x))$. The induction hypothesis shows that there is an exact sequence $R^{n-1} \xrightarrow{A} R^{n-1} \to Y \to 0$ with $A = \begin{pmatrix} x & c_{1,2} & \cdots & c_{1,n-1} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & c_{n-2,n-1} \\ 0 & \cdots & 0 & x \end{pmatrix}$ such that $\begin{pmatrix} c_{1,j} \\ \vdots \\ c_{j-1,j} \end{pmatrix} (0:x) \subseteq \operatorname{Im} \begin{pmatrix} x & c_{1,2} & \cdots & c_{1,j-1} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & c_{j-2,j-1} \\ 0 & \cdots & 0 & x \end{pmatrix}$ for all $2 \leq j \leq n-1$. We have a commutative diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & (0:x) \arrow[d] & \\
0 \arrow[r] & R^{n-1} \arrow[r, "\binom{1}{0}"] \arrow[d, "A"'] & R^{n-1} \oplus R \arrow[r, "(0\ 1)"] \arrow[d, "\left(\begin{smallmatrix} A & B \\ 0 & x \end{smallmatrix}\right)"'] & R \arrow[r] \arrow[d, "x"'] & 0 \\
0 \arrow[r] & R^{n-1} \arrow[r, "\binom{1}{0}"] \arrow[d, "f"'] & R^{n-1} \oplus R \arrow[r, "(0\ 1)"] \arrow[d] & R \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Y \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & R/(x) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

with exact rows and columns. The induced map $g : (0:x) \to Y$ is the zero map by the snake lemma. By diagram chasing, we see that $g(r) = f(Br)$ holds for each $r \in (0:x)$. Hence we have $f(Br) = 0$ for all $r \in (0:x)$, whence Br is in the image of the map $A : R^{n-1} \to R^{n-1}$. Writing $B = \begin{pmatrix} c_{1,n} \\ \vdots \\ c_{n-1,n} \end{pmatrix}$, we obtain an inclusion relation $\begin{pmatrix} c_{1,n} \\ \vdots \\ c_{n-1,n} \end{pmatrix} (0:x) \subseteq \operatorname{Im} \begin{pmatrix} x & c_{1,2} & \cdots & c_{1,n-1} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & c_{n-2,n-1} \\ 0 & \cdots & 0 & x \end{pmatrix}$. Consequently, we have

$$\begin{pmatrix} c_{1,j} \\ \vdots \\ c_{j-1,j} \end{pmatrix} (0:x) \subseteq \operatorname{Im} \begin{pmatrix} x & c_{1,2} & \cdots & c_{1,j-1} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & c_{j-2,j-1} \\ 0 & \cdots & 0 & x \end{pmatrix}$$

for all $2 \leq j \leq n$. The middle column of the above diagram gives an exact sequence

$$R^n \xrightarrow{\begin{pmatrix} x & c_{1,2} & \cdots & c_{1,n-1} & c_{1,n} \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & c_{n-2,n-1} & c_{n-2,n} \\ 0 & \cdots & 0 & x & c_{n-1,n} \\ 0 & \cdots & 0 & 0 & x \end{pmatrix}} R^n \longrightarrow M \longrightarrow 0.$$

Thus the proof of the proposition is completed. □

Now we can prove the following result concerning the structure of $\mathsf{ext}_R(R/(x))$, which is the main result of this paper.

Theorem 2.6. *Let x, y be part of a minimal system of generators of $\mathfrak{m}$ with $xy = 0$. Then k does not belong to $\mathsf{ext}_R(R/(x))$.*

Proof. Let e be the embedding dimension of R. We have $e \geq 2$, and write $\mathfrak{m} = (x, y, z_3, \dots, z_e)$. Let us assume that k belongs to $\mathsf{ext}_R(R/(x))$. We want to derive a contradiction. By Proposition 2.4, the module k is isomorphic to a direct summand of a module $M \in \mathsf{filt}_R(R/(x))$. We have an isomorphism $M \cong k \oplus N$ for some R-module N, and M belongs to $\mathsf{filt}_R^n(R/(x))$ for some $n \geq 1$. Proposition 2.5 gives an exact sequence

$$R^n \xrightarrow{\begin{pmatrix} x & c_{1,2} & \cdots & c_{1,n} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & c_{n-1,n} \\ 0 & \cdots & 0 & x \end{pmatrix}} R^n \longrightarrow M \longrightarrow 0 \tag{2.6.1}$$

of R-modules such that

$$\begin{pmatrix} c_{1,j} \\ \vdots \\ c_{j-1,j} \end{pmatrix} (0 : x) \subseteq \operatorname{Im} \begin{pmatrix} x & c_{1,2} & \cdots & c_{1,j-1} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & c_{j-2,j-1} \\ 0 & \cdots & 0 & x \end{pmatrix}$$

for all $2 \leq j \leq n$. Since y is in $(0 : x)$, there are elements $d_{1,j}, \dots, d_{j-1,j} \in R$ such that

$$\begin{pmatrix} c_{1,j}y \\ \vdots \\ c_{j-1,j}y \end{pmatrix} = \begin{pmatrix} x & c_{1,2} & \cdots & c_{1,j-1} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & c_{j-2,j-1} \\ 0 & \cdots & 0 & x \end{pmatrix} \begin{pmatrix} d_{1,j} \\ \vdots \\ d_{j-1,j} \end{pmatrix}.$$

Hence the equality

$$c_{i,j}y = xd_{i,j} + c_{i,i+1}d_{i+1,j} + \cdots + c_{i,j-1}d_{j-1,j}$$

holds for $2 \leq j \leq n$ and $1 \leq i \leq j - 1$.

We claim that the elements $c_{i,j}, d_{i,j}$ belong to $\mathfrak{m}$ for all $2 \leq j \leq n$ and $1 \leq i \leq j-1$. Indeed, the hypothesis of induction on j implies that $c_{i,l}$ is in $\mathfrak{m}$ for $i+1 \leq l \leq j-1$, and the assumption of descending induction on i shows that $d_{l,j}$ is in $\mathfrak{m}$ for $i+1 \leq l \leq j-1$. Hence we have $c_{i,j}y - xd_{i,j} \in \mathfrak{m}^2$, which gives an equality

$$\overline{c_{i,j}} \cdot \overline{y} - \overline{x} \cdot \overline{d_{i,j}} = \overline{0}$$

in $\mathfrak{m}/\mathfrak{m}^2$. Since $\overline{x}, \overline{y}$ are part of a k-basis of $\mathfrak{m}/\mathfrak{m}^2$, we have $\overline{c_{i,j}} = \overline{d_{i,j}} = \overline{0}$ in k. Therefore, $c_{i,j}, d_{i,j}$ belong to $\mathfrak{m}$, as desired.

By elementary column operations, the matrix $\begin{pmatrix} x & c_{1,2} & \cdots & c_{1,n} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & c_{n-1,n} \\ 0 & \cdots & 0 & x \end{pmatrix}$ can be transformed into a matrix $\begin{pmatrix} x & b_{1,2} & \cdots & b_{1,n} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & b_{n-1,n} \\ 0 & \cdots & 0 & x \end{pmatrix}$ such that each $b_{i,j}$ is an element of the ideal $I = (y, z_3, \dots, z_e)$. We have an exact sequence

$$R^n \xrightarrow{\begin{pmatrix} x & b_{1,2} & \cdots & b_{1,n} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & b_{n-1,n} \\ 0 & \cdots & 0 & x \end{pmatrix}} R^n \longrightarrow M \longrightarrow 0,$$

and applying $- \otimes_R R/I$ to this, we get an exact sequence

$$(R/I)^n \xrightarrow{\begin{pmatrix} x & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x \end{pmatrix}} (R/I)^n \longrightarrow M/IM \longrightarrow 0.$$

Hence we have an isomorphism $M/IM \cong (R/I + (x))^n = k^n$. Since $M/IM \cong k \oplus N/IN$, we see that N/IN is isomorphic to k^{n-1}, and get an equality

$$\beta_1^{R/I}(N/IN) = (n-1)\beta_1^{R/I}(k) \tag{2.6.2}$$

of Betti numbers. There is an exact sequence $R^{\beta_1^R(N)} \to R^{\beta_0^R(N)} \to N \to 0$ of R-modules, and tensoring R/I with this gives an exact sequence $(R/I)^{\beta_1^R(N)} \to (R/I)^{\beta_0^R(N)} \to N/IN \to 0$ of R/I-modules. It follows from this that

$$\beta_1^{R/I}(N/IN) \le \beta_1^R(N). \tag{2.6.3}$$

The isomorphism $M \cong k \oplus N$ shows

$$\beta_1^R(M) = \beta_1^R(k) + \beta_1^R(N) = e + \beta_1^R(N). \tag{2.6.4}$$

The existence of the exact sequence (2.6.1) implies

$$\beta_1^R(M) \le n. \tag{2.6.5}$$

Since $\mathfrak{m}/I = x(R/I)$ and $x \notin I$, we have

$$\beta_1^{R/I}(k) = 1. \tag{2.6.6}$$

Using the (in)equalities (2.6.2)–(2.6.6), we obtain

$$n - 1 = (n-1)\beta_1^{R/I}(k) = \beta_1^{R/I}(N/IN) \le \beta_1^R(N) = \beta_1^R(M) - e \le n - e,$$

whence $e \le 1$. This is a desired contradiction; this contradiction completes the proof of the theorem. $\square$

Let R be an Artinian local ring. Then, using the fact that every R-module M is annihilated by the ideal $\mathfrak{m}^{\ell(M)}$, we can check that the equality $\mathfrak{m}^{\ell(R) - \operatorname{edim} R + 1} = 0$ holds. (Here, $\ell(M)$ and $\operatorname{edim} R$ denote the length of M and the embedding dimension of R,

respectively.) Recall that R is called *stretched* if $\mathfrak{m}^i \neq 0$ for all $i < \ell(R) - \operatorname{edim} R + 1$, or equivalently, if $\mathfrak{m}^{\ell(R)-\operatorname{edim} R} \neq 0$.

Example 2.7. (1) Every Artinian Gorenstein local ring R with $\mathfrak{m}^3 = 0$ that is not a field is stretched.

(2) Let k be a field, and let

$$R = k[[x, y, z]]/(xy, xz, yz, x^3 - y^2, x^3 - z^2)$$

be a residue ring of a formal power series ring over k. Then R is an Artinian Gorenstein local ring. Since $\ell(R) = 6$, $\operatorname{edim} R = 3$ and $\mathfrak{m}^3 = (x^3) \neq 0$, the ring R is stretched.

Now we have a sufficient condition for $\operatorname{\mathsf{mod}} R$ to have a nontrivial extension-closed subcategory.

Corollary 2.8. *Let R be a stretched Artinian Gorenstein local ring with* $\operatorname{edim} R \geq 2$. *Then* $\operatorname{\mathsf{mod}} R$ *has a nontrivial extension-closed subcategory.*

Proof. If $\operatorname{edim} R < \ell(R) - 2$, then by [5, Theorem 1.1] there exist elements $x, y \in R$ with $xy = 0$ which form part of minimal system of generators of $\mathfrak{m}$, and Theorem 2.6 shows that $\mathsf{ext}_R(R/(x))$ is a nontrivial extension-closed subcategory of $\operatorname{\mathsf{mod}} R$.

Let $\operatorname{edim} R \geq \ell(R) - 2$. Then we have $\mathfrak{m}^3 = 0$. Take an element $x \in \mathfrak{m} \setminus \mathfrak{m}^2$.

First, assume that $(0 : x)$ is not contained in $(x) + \mathfrak{m}^2$. Then there exists an element $y \in (0 : x)$ which does not belong to $(x) + \mathfrak{m}^2$, and we see that $\overline{x}, \overline{y}$ form part of a k-basis of $\mathfrak{m}/\mathfrak{m}^2$. Hence x, y are part of a minimal system of generators of $\mathfrak{m}$ with $xy = 0$, and the assertion follows from Theorem 2.6.

Next, assume that $(0 : x)$ is contained in $(x) + \mathfrak{m}^2$. Then we have

$$(x) \overset{\text{(a)}}{=} (0 : (0 : x)) \supseteq (0 : (x) + \mathfrak{m}^2) = (0 : x) \cap (0 : \mathfrak{m}^2) \overset{\text{(b)}}{=} (0 : x).$$

Here, the equality (a) follows from the double annihilator property (cf. [4, Exercise 3.2.15]), and (b) from the inclusion $(0 : \mathfrak{m}^2) \supseteq \mathfrak{m}$. Suppose that $(0 : x) \neq (x)$. Then we have $x\mathfrak{m} \subseteq \mathfrak{m}^2 \subseteq (0 : x) \subsetneq (x)$ and $\ell_R((x)/x\mathfrak{m}) = 1$, which imply $x\mathfrak{m} = \mathfrak{m}^2 = (0 : x)$. Hence $\mathfrak{m} \subseteq (0 : \mathfrak{m}^2) = (0 : (0 : x)) = (x)$, which contradicts the assumption that $\operatorname{edim} R \geq 2$. Thus the equality $(0 : x) = (x)$ holds, and there exists an exact sequence

$$\cdots \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} R \to R/(x) \to 0$$

of R-modules. This implies that $R/(x)$ belongs to the subcategory $\mathcal{X}$ of $\operatorname{\mathsf{mod}} R$ consisting of all R-modules with bounded Betti numbers, which is extension-closed. Hence $\mathcal{X}$ is neither $\mathbf{0}$ nor $\operatorname{\mathsf{add}} R$, and we also have $\mathcal{X} \neq \operatorname{\mathsf{mod}} R$ because R is not a hypersurface by the assumption that $\operatorname{edim} R \geq 2$ again. Therefore $\mathcal{X}$ is a nontrivial extension-closed subcategory of $\operatorname{\mathsf{mod}} R$. $\square$

We can guarantee that our Conjecture 1.4 holds true for a stretched Artinian Gorenstein local ring. The following result follows from Proposition 1.3 and Corollary 2.8.

Corollary 2.9. *Let R be a stretched Artinian Gorenstein local ring. Then the following are equivalent.*

(1) *R is an Artinian hypersurface.*

(2) $\mathsf{mod}\, R$ *has only trivial extension-closed subcategories.*

We end this paper by posing a question.

Question 2.10. An extension-closed subcategory of $\mathsf{mod}\, R$ is called *resolving* if it contains R and is closed under syzygies. Does the assumption of Theorem 2.6 imply that k does not belong to the smallest resolving subcategory of $\mathsf{mod}\, R$ containing $R/(x)$?

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References

[1] L. L. Avramov, Infinite free resolutions, *Six lectures on commutative algebra (Bellaterra, 1996)*, 1–118, Progr. Math., 166, *Birkhäuser, Basel*, 1998.
[2] L. L. Avramov; V. N. Gasharov; I. V. Peeva, Complete intersection dimension, *Inst. Hautes Études Sci. Publ. Math.* No. 86 (1997), 67–114 (1998).
[3] P. A. Bergh, On complexes of finite complete intersection dimension, *Homology, Homotopy Appl.* **11** (2009), no. 2, 49–54.
[4] W. Bruns; J. Herzog, Cohen-Macaulay rings, revised edition, Cambridge Studies in Advanced Mathematics, 39, *Cambridge University Press, Cambridge*, 1998.
[5] J. D. Sally, Stretched Gorenstein rings, *J. London Math. Soc. (2)* **20** (1979), no. 1, 19–26.
[6] R. Takahashi, Classifying thick subcategories of the stable category of Cohen-Macaulay modules, *Adv. Math.* (to appear), `http://arxiv.org/abs/0908.0107`.
[7] J. Tate, Homology of Noetherian rings and local rings, *Illinois J. Math.* **1** (1957), 14–27.

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